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Chapter 7: Rational and Irrational Numbers (3 weeks)

Utah Core Standard(s):

- Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational. (8.EE.2)
- Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers, show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number. (8.NS.1)
- Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g.,π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$, is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations. (8.NS.2)

Academic Vocabulary: square, perfect square, square root, $\sqrt{}$, cube, perfect cube, cube root, $\sqrt[3]{}$, decimal expansion, repeating decimal, terminating decimal, rational number, irrational number, truncate, decimal approximation, real number, real number line

Chapter Overview:
Up to this point, students have been gradually building an understanding of the rational number system. In 8th grade, students extend their knowledge of the number system even further to include irrational numbers. With the introduction of irrational numbers, students have now built an understanding of the real number system, those numbers that can be associated to a point on the real number line. Using tilted squares, students explore the meaning of square root and construct physical lengths of irrational numbers. They transfer these lengths to a number line and realize that even though we can construct the physical length of an irrational number and approximate its value, we cannot give an exact numerical value of an irrational number. As students work through the definition of an irrational number, they further solidify their understanding of rational numbers. Later in the chapter, students learn methods for approximating the numerical value of irrational numbers to desired degrees of accuracy, estimate the value of expressions containing irrational numbers, and compare and order rational and irrational numbers.
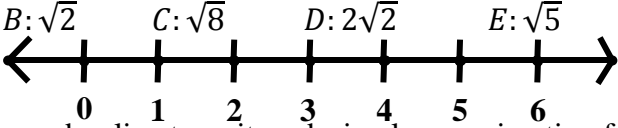
Connections to Content:
Prior Knowledge: Students have worked a great deal with rational numbers up to this point. They have defined and worked with the subsets of rational numbers. They have represented rational numbers on a number line, expressed rational numbers in different but equivalent forms, and operated with rational numbers. Students have also worked a great deal with slope, have an understanding of area, and know how to find the area of polygons and irregular shapes which will help them to access the tilted square material.

Future Knowledge: In subsequent courses, students will continue to extend their knowledge of the number system even further. For example, students will learn about complex numbers as a way to solve quadratic equations that have a negative discriminant. They will also continue to work with irrational numbers, learning how to operate on irrational numbers.

MATHEMATICAL PRACTICE STANDARDS (emphasized):

	Make sense of problems and persevere in solving them.	A hospital has asked a medical supply company to manufacture intravenous tubing (IV tubing) that has a minimum opening of 7 square millimeters and a maximum opening of 7.1 square millimeters for the rapid infusion of fluids. The medical design team concludes that the radius of the tube opening should be 1.5 mm. Two supervisors review the design team's plans, each using a different estimation for π. **Supervisor 1:** Uses 3 as an estimation for π **Supervisor 2:** Uses 3.1 as an estimation for π The supervisors tell the design team that their designs will not work. The design team stands by their plans and tells the supervisors they are wrong. Who is correct and why? Recall that the formula for the area of a circle is $A = \pi r^2$. *In this problem, students realize the effects of approximating the value of irrational numbers. They must decide which estimation of π is appropriate for the given situation, appreciating that the precision of the estimation may have profound impact on decisions people make in the real world.*
	Construct viable arguments and critique the reasoning of others.	**Directions:** The table below contains statements about rational and irrational numbers. If the statement is true, put a check in the box. If the statement is not true, write a correct statement. • You can always use a calculator to determine whether a number is rational or irrational by looking at its decimal expansion. • The number 0.256425642564 ... is rational. • You can build a perfect cube with 36 unit cubes. • If you divide an irrational number by 2, you will still have an irrational number. *Students must have a clear understanding of rational and irrational numbers to assess whether the statements are true or false. If the statement is flawed, students must identify the flaw, and construct a statement that is true. Due to the fact that there are several possible ways to change the statements to make them true, students must communicate their statements to classmates, justify the statements, and question and respond to the statements made by others.*
	Look for and express regularity in repeated reasoning.	Change the following rational numbers into decimals **without** the use of a calculator. $\frac{1}{7}$ *This problem allows students to understand why the decimal expansion of a rational number either always terminates or repeats a pattern. Working through this problem, and others, students begin to understand that eventually the pattern must repeat because there are only so many ways that the algorithm can go. Once a remainder repeats itself in the division process, the decimal expansion will start to take on a repeating pattern. Students should see this when they begin repeating the same calculations over and over again and conclude they have a repeating decimal.*

|	**Attend to Precision**	Use the following approximations and calculations to answer the questions below. Do not use a calculator. Approximation: π is between 3.14 and 3.15 Calculations: $3.1^2 = 9.61$ $3.2^2 = 10.24$ $3.16^2 = 9.9856$ $3.17^2 = 10.0489$ Put the following numbers in order from **least to greatest**. $\sqrt{10}, 3\frac{1}{10}, 3.\overline{1}, \pi$, side length of a square with an area of 9 Find a number between $3\frac{1}{10}$ and $3.\overline{1}$. Find a number between 3.1 and $\sqrt{10}$. *This task demands mastery of the topics learned in the chapter. Students must have a very clear understanding of square roots, repeating decimals, and irrational numbers. They must closely analyze the decimal expansions (approximations) of the numbers as well as the calculations given to be able to compare and order the numbers.*
	Use appropriate tools strategically.	**Directions:** Show the length of the following numbers on the number line below. Use the grid on the following page to construct lengths where needed and transfer those lengths onto the number line. Then answer the questions that follow. **Note:** On the grid, a horizontal or vertical segment joining two dots has a length of 1. On the number line, the unit length is the same as the unit length on the dot grid. $A: \sqrt{25}$ $B: \sqrt{2}$ $C: \sqrt{8}$ $D: 2\sqrt{2}$ $E: \sqrt{5}$ $F: 2\sqrt{5}$ 0 1 2 3 4 5 6 1. Use the number line to write a decimal approximation for $\sqrt{2}$. 2. Would 1.41 be located to the right or to the left of $\sqrt{2}$ on the number line? 3. Describe and show how you can put $-\sqrt{2}$ on the number line. Estimate the value of this expression. 4. Describe and show how you can put $(2 + \sqrt{2})$ on the number line. Estimate the value of this expression. 5. Describe and show how you can put $(2 - \sqrt{2})$ on the number line. Estimate the value of this expression. 6. Describe and show how you can put $2\sqrt{2}$ on the number line. Estimate the value of this expression. *To solve this problem, students use dot paper to construct physical lengths of irrational numbers. They can then transfer these segments to the number line using patty (or tracing) paper. Once on the number line, students can use these tools (number line, dot paper, patty paper, constructed segments) to approximate the value of given expressions (i.e. $(2 + \sqrt{2})$.*

	Look for and make use of structure	Square A shown below has an area of 8 square units. Determine the following measures:
		
		   a. The area of one of the smaller squares that makes up Square A
		
		   b. The side length of one of the smaller squares that makes up Square A
		
		   c. The side length of the large square A (written 2 different ways)
		
		This problem allows students to use structure to understand why the $\sqrt{8}$ is the same as $2\sqrt{2}$. They can see the equivalence in the concrete model. A square with an area of 8 (see Square A) has a side length of $\sqrt{8}$ units. This side length is comprised of 2 smaller, congruent segments that each measure $\sqrt{2}$ units as they are each the side length of a square with an area of 2. This concrete representation builds a conceptual understanding for students as we then move to the algorithm for simplifying square roots.

	Reason abstractly and quantitatively.	The decimal $0.\overline{3}$ is a repeating decimal that can be thought of as $0.33333\ldots$ where the "…" indicates that the 3s repeat forever. If they repeat forever, how can we write this number as a fraction? Here's a trick that will eliminate our repeating 3s. *To solve this problem, students create and solve a system of linear equations. The skills and knowledge they learned about systems of equations become an abstract tool that allows students to write repeating decimals as fractions, proving that they do in fact fit the definition of a rational number.*

7.0 Anchor Problem: Zooming in on the Number Line

Directions: Place the following sets of numbers on the number lines provided. You will have to specify where 0 is on each number line and decide on the length of each interval that makes sense for a given problem. Label each point.

$A: 3$ $B: 4$ $C: 3.5$ $D: -4$ $E: -5$ $F: -4.5$
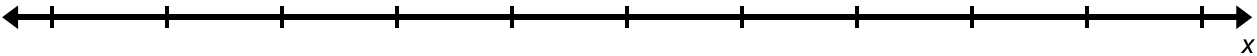
$R: \frac{1}{3}$ $S: \frac{2}{3}$ $T: \frac{5}{3}$ $U: 2\frac{2}{3}$ $V: \frac{6}{3}$
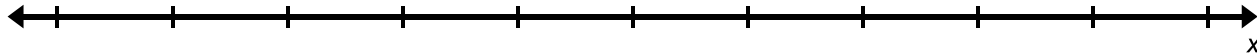
$L: -\frac{1}{4}$ $M: \frac{3}{4}$ $N: -1\frac{1}{2}$ $O: 1.75$ $P: -2$
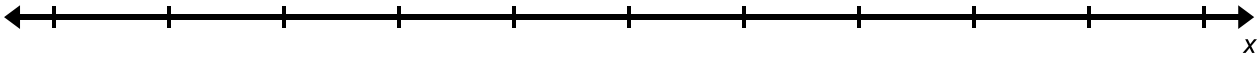
$V: \frac{1}{10}$ $W: \frac{3}{10}$ $X: \frac{1}{2}$ $Y: \frac{9}{10}$ $Z: \frac{10}{10}$
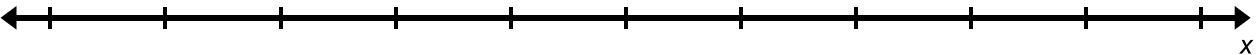
$H: 0.1$ $I: 0.2$ $J: 0.15$ $K: 0.11$ $L: 0.101$
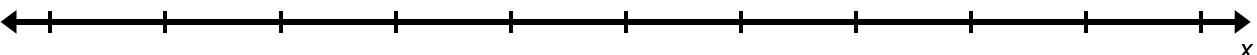
$A: 3.1$ $B: 3.2$ $C: 3.15$ $D: 3.11$ $E: 3.105$ $F: 3.111$
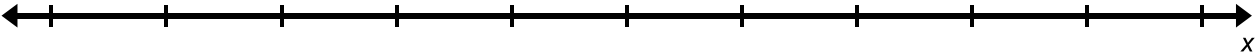
Directions: Refer to the last number line on the previous page to answer the questions that follow.

1. Are there other numbers you can place between 3.1 and 3.11? If yes, what is a number?

2. Are there other numbers you can place between 3.11 and 3.111? If yes, what is a number?

3. How are you coming up with the numbers? Are there others? How do you know?

4. Where would you put $3.\overline{1}$ on the number line and why?

5. What can you conclude about the real number line based on this activity?

Section 7.1: Represent Numbers Geometrically

Section Overview:
In this section, students are exposed to a "new" kind of number. This chapter starts with a review of background knowledge – finding the area of polygons and irregular shapes, using ideas of slope to create segments of equal length, and reviewing the definition of a square. Then students build squares with different areas and determine the side length of these squares. Students realize that we can create squares that are not perfect; however we cannot find an exact numerical value for the side length of these squares. During the process, students gain an understanding of what is meant by the square root of a number and begin to develop an understanding of irrational numbers. We see that we can construct the length of an irrational number, specifically a non-perfect square, and transfer it to a number line but we cannot find the exact value of these numbers. Students also simplify square roots, connecting the simplified answer to a physical model. Using cubes and volume, students gain an understanding of what is meant by the cube root of a number and evaluate small perfect cubes.

Concepts and Skills to Master:
By the end of this section, students should be able to:

1. Understand the relationship between the side length of a square and its area.
2. Understand the relationship between the side length of a cube and its volume.
3. Understand what is meant by the square root and cube root symbols.
4. Construct physical lengths of perfect and non-perfect squares.
5. Evaluate the square roots of small perfect squares and the cube roots of small perfect cubes.
6. Simplify square roots.

7.1a Class Activity: Background Knowledge

<u>Activity 1: Finding Area of Irregular Shapes</u>
Directions: Find the area of the following shapes. On the grid, a horizontal or vertical segment joining two dots has a length of 1. Put your answers on the lines provided below the grid.
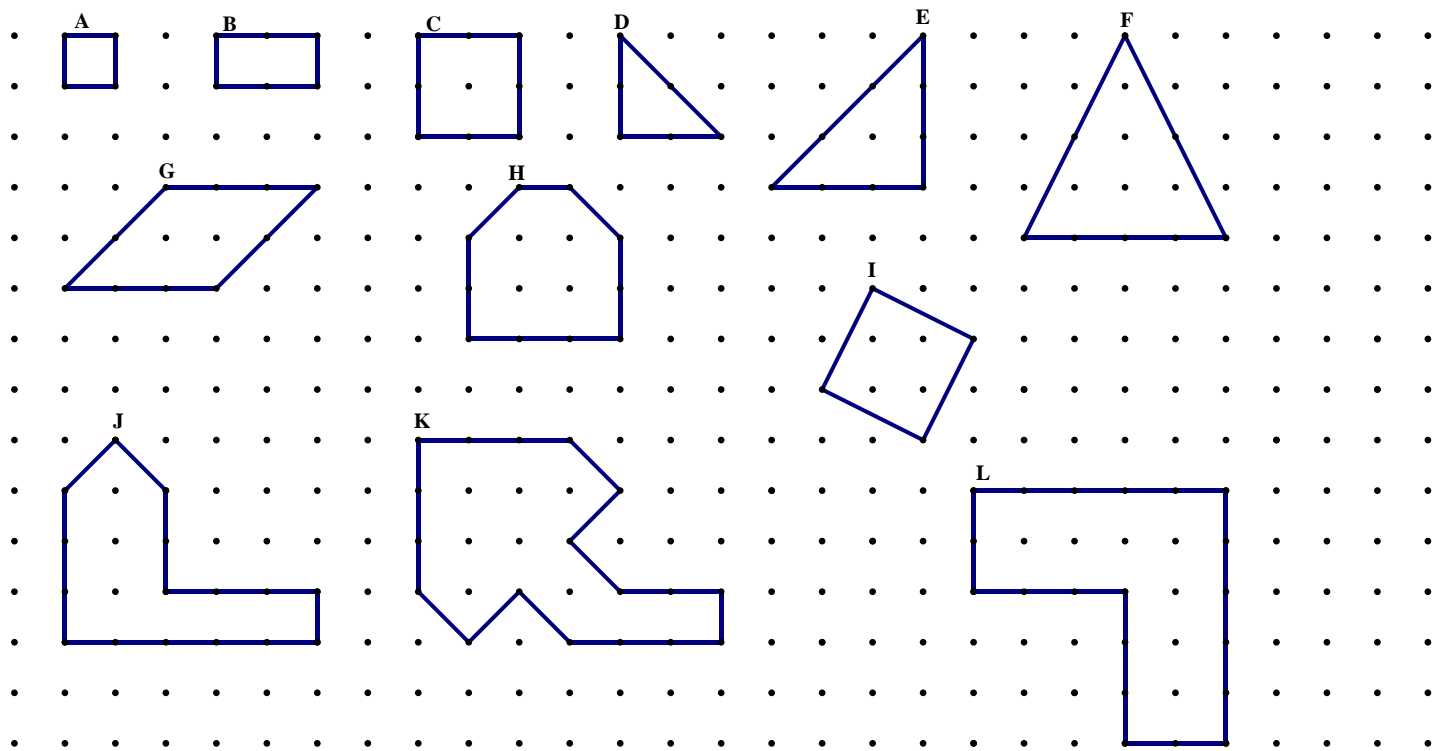
A: _____ B: _____ C: _____ D: _____ E: _____ F: _____

G: _____ H: _____ I: _____ J: _____ K: _____ L: _____

Directions: Use a different method than used above to find the areas of the shapes below.

D: _____ G: _____ H: _____ J: _____ L: _____

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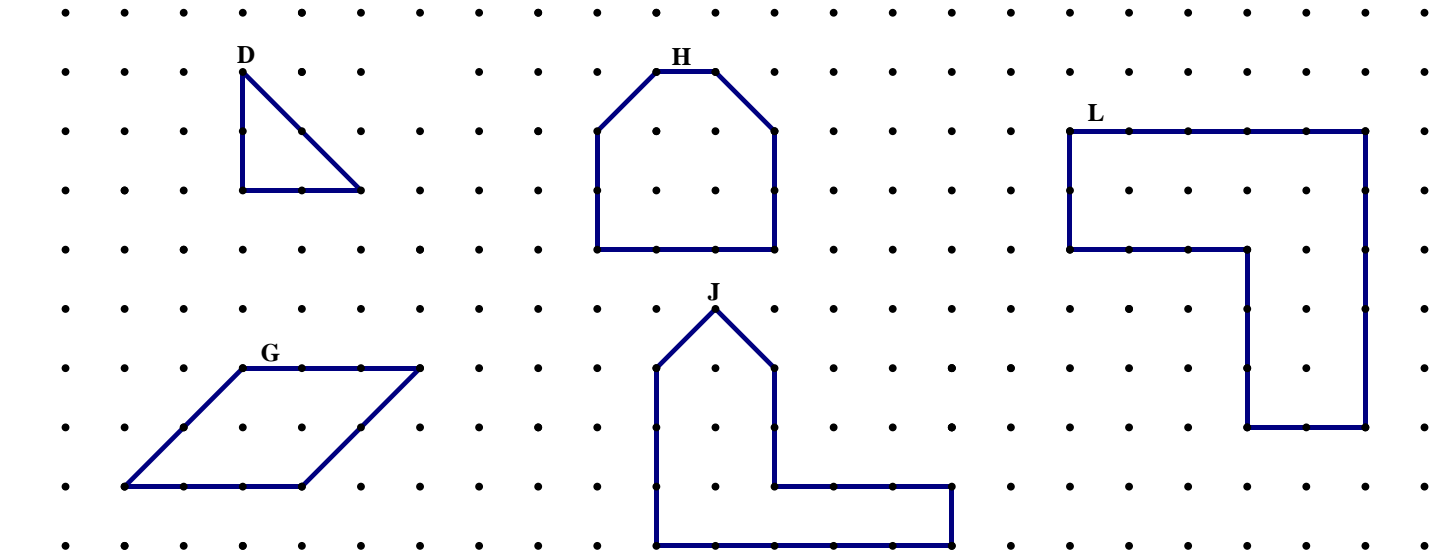
Activity 2: Slopes and Lengths of Segments

1. In the space to the right of the dot grid, write down observations you have about the line segments shown on the grid.
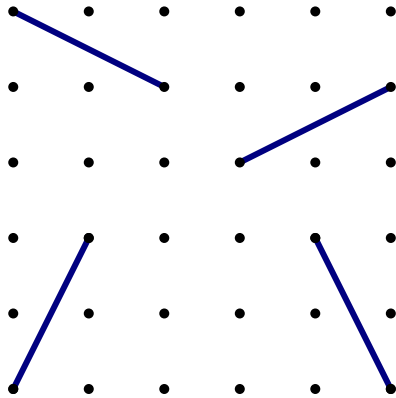
2. Using the ideas from the previous problem, construct 3 additional segments that are the same length as the segment shown below. All of your segments must have a different orientation as in the previous problem.
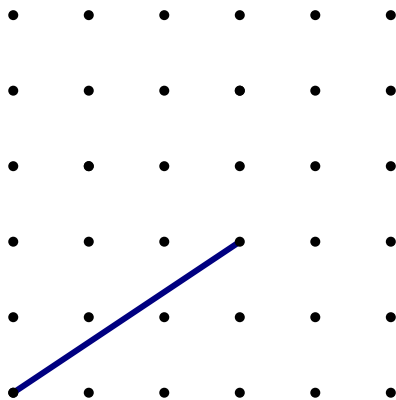
7.1a Homework: Background Knowledge

1. Find the areas of the following shapes. On the grid, a horizontal or vertical segment joining two dots has a length of 1. Put your answers on the lines provided below the grid.
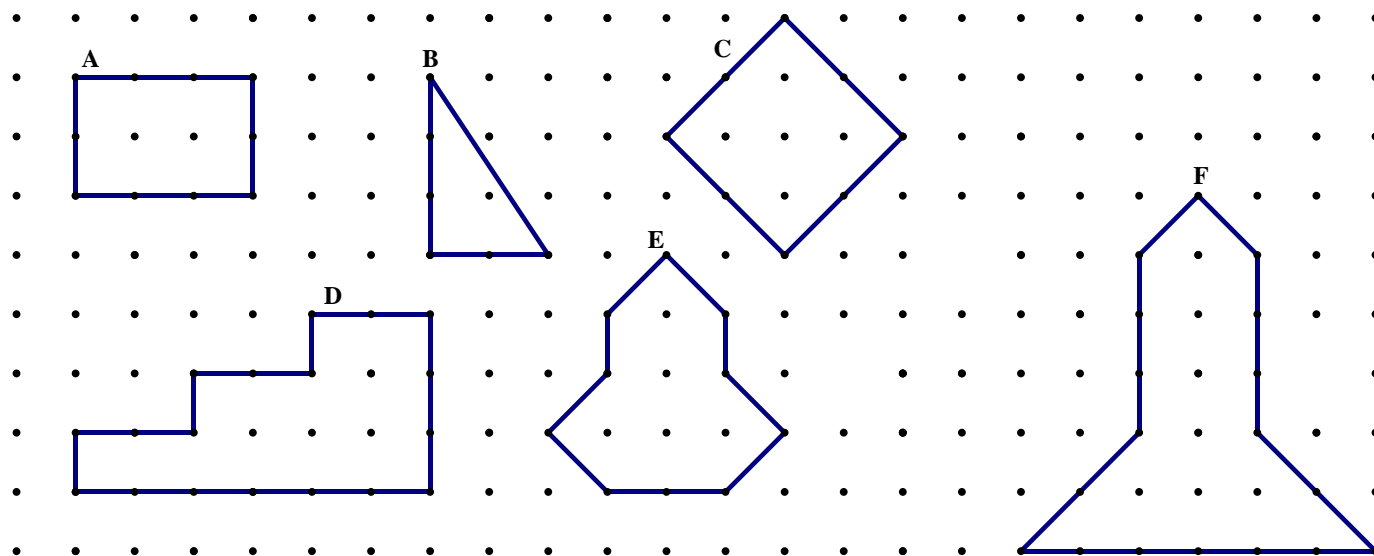
A: _____ B: _____ C: _____ D: _____ E: _____ F: _____

2. Show a second method for finding the area of shape C.

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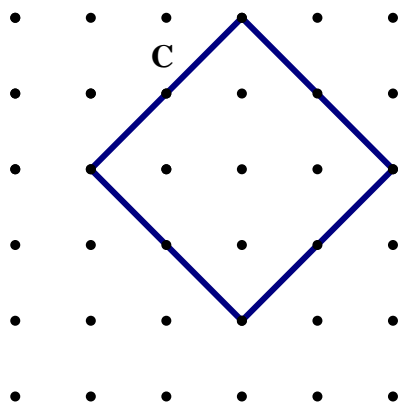
7.1b Class Activity: Squares, Squares, and More Squares

On the following pages of dot paper:
1) Create as many different squares with areas from 1 - 100 as possible. On the grid, a horizontal or vertical segment joining two dots has a length of 1. Each corner of the square must be on a dot.
2) Find the area of each square you made and label each square with its area.
3) Complete the table below using the squares you created above.

Area	Side Length
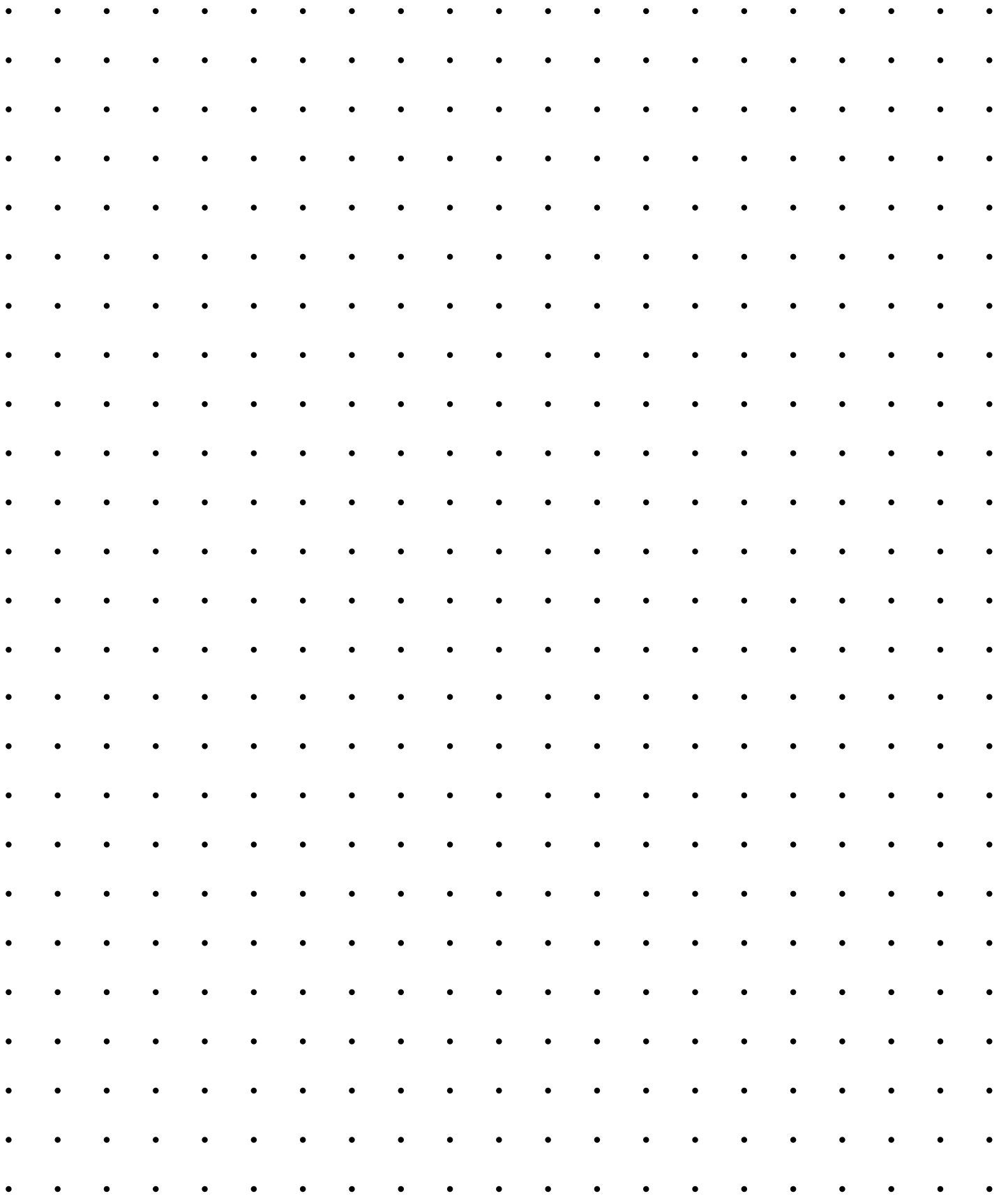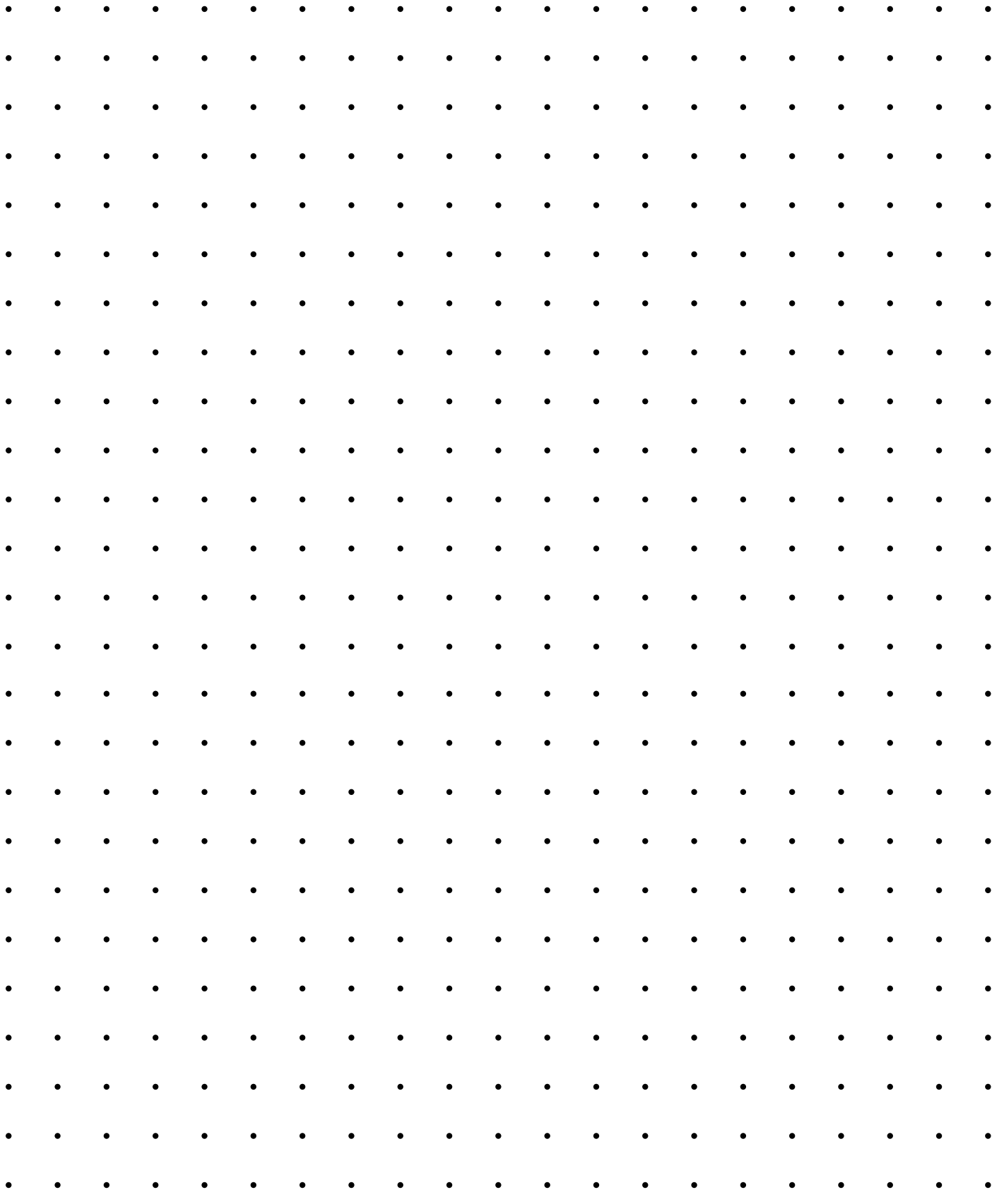
<u>Summarize Learning</u>

1. Complete the following table…

Area (square units)	Length of Side (units)
1	
9	
	5
2	
5	
	$\sqrt{13}$
	$\sqrt{5}$
100	
	20

Directions: Complete the following sentences. Provide examples to support your statements.

2. Any whole number (1, 2, 3, 4, etc.) raised to the second power…

3. To find the area of a square given the side length of the square…

4. To find the side length of a square given the area of the square…

5. Simplify the following.
   a. $\sqrt{36}$ d. $\sqrt{1}$

   b. $\sqrt{121}$ e. $\sqrt{100}$

   c. $\sqrt{16}$ f. $\sqrt{49}$

7.1b Homework: Squares, Squares, and More Squares

1. List the first 12 perfect square numbers.

2. What is the side length of a square with an area of 9 units^2?

3. What is the area of a square with a side length of 2 units?

4. Complete the following table.

Area (square units)	Length of Side (units)
4	
49	
144	
	10
2	
8	
	$\sqrt{15}$
	$\sqrt{41}$
	1
	9

5. Simplify the following:
   a. $\sqrt{9}$

   b. $\sqrt{100}$

   c. $\sqrt{64}$

   d. $\sqrt{4}$

   e. $\sqrt{144}$

   f. $\sqrt{81}$

7.1c Class Activity: Squares, Squares, and More Squares Cont.

1. Determine the lengths of line segments a through l without the use of a ruler. Write your answers in the space provided below each grid.

a. _____

b. _____

c. _____

d. _____

e. _____

f. _____

g. _____

h. _____

i. _____

j. _____

k. _____

l. _____
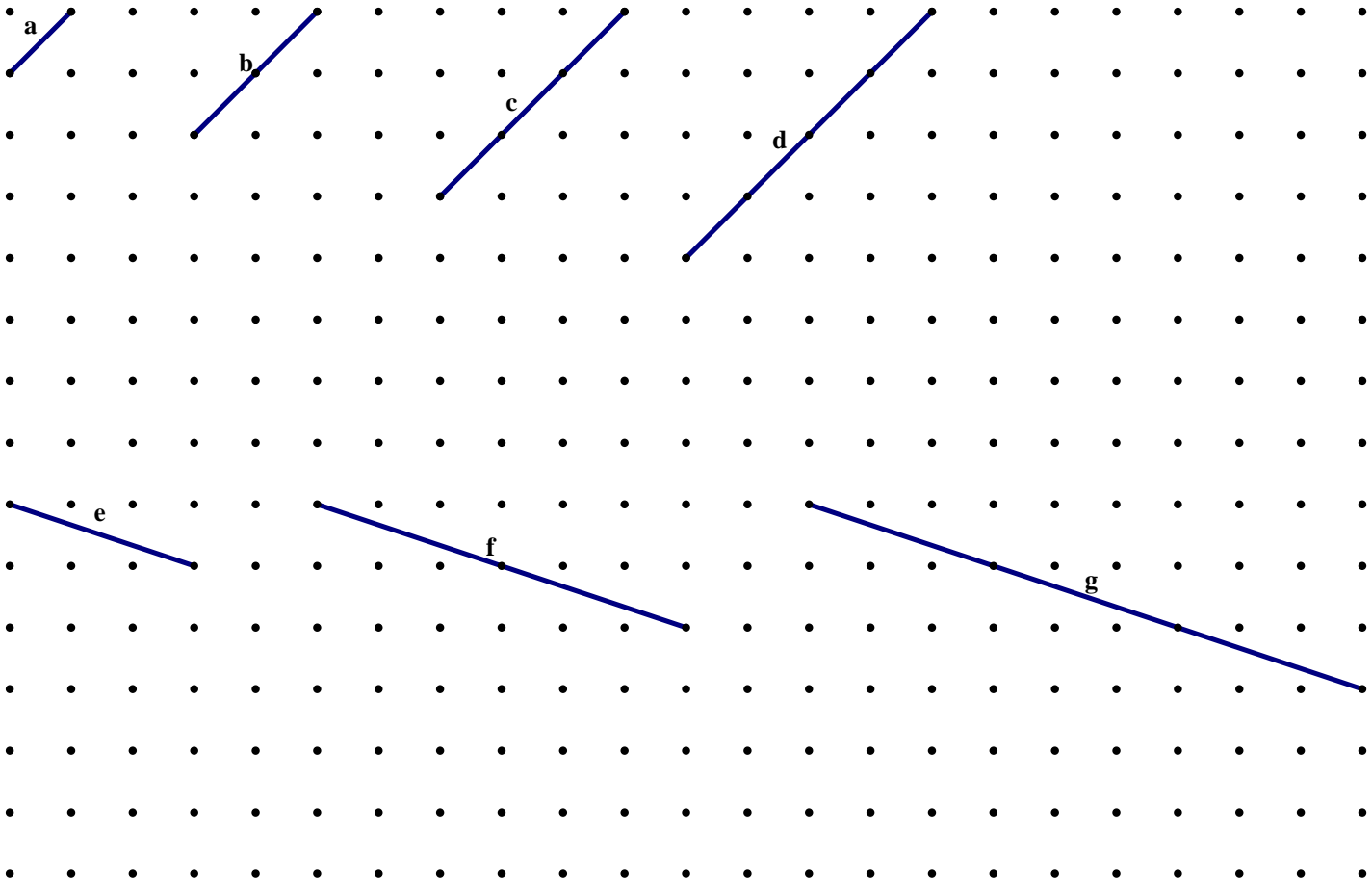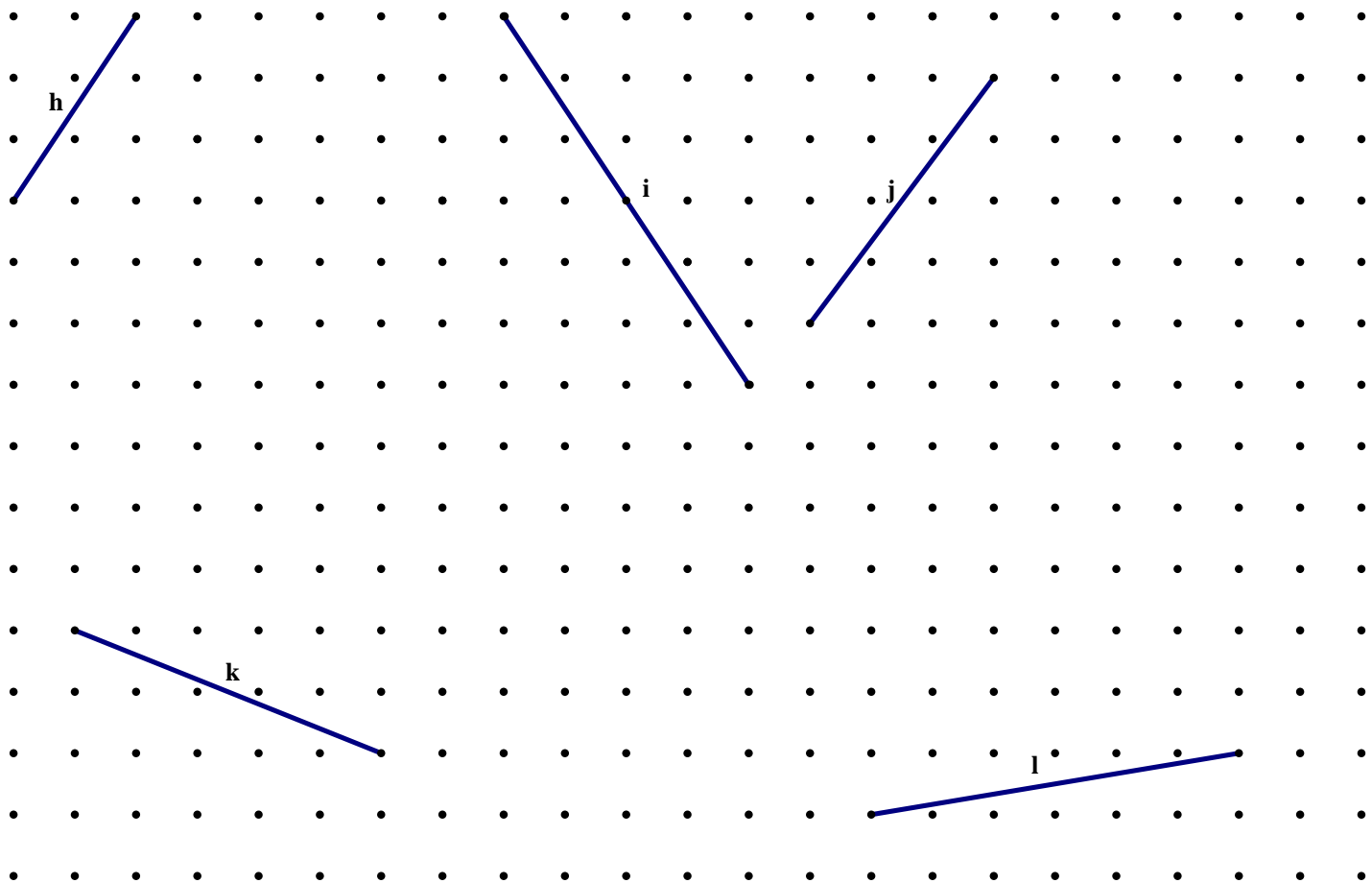
2. On the grid below, construct a segment with a length of $\sqrt{5}$ units. Explain how you know your segment measures $\sqrt{5}$ units.

7.1d Classwork: Simplifying Square Roots

In the previous sections, we have learned how to simplify square roots of perfect squares. For example, we know that $\sqrt{36} = 6$. What about the square roots of non-perfect squares? How do we know that they are in simplest form? For example, is $\sqrt{5}$ in simplest form? How about $\sqrt{8}$? $\sqrt{147}$? Let's take a look.

Directions: Use the squares on the grid below to answer the questions that follow. Each of the large squares A, B, and C has been cut into four smaller squares of equal size.
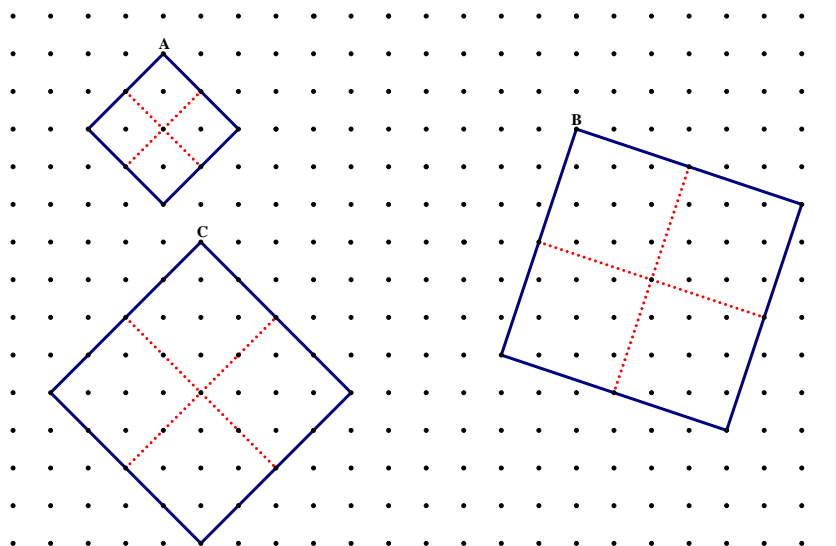
1. Square A has an area of 8 square units. Answer the following questions.
   a. What is the area of one of the smaller squares that makes up Square A? _____

   b. What is the side length of one of the smaller squares that makes up Square A? _____

   c. What is the side length of the large square A (written 2 different ways)? _____

2. Square B has an area of 40 square units. Answer the following questions.
   a. What is the area of one of the smaller squares that makes up Square B? _____

   b. What is the side length of one of the smaller squares that makes up Square B? _____

   c. What is the side length of the large square B (written two different ways)? _____

3. Square C has an area of 32 square units. Answer the following questions.
   a. What is the area of one of the smaller squares that makes up Square C? _____

   b. What is the side length of one of the smaller squares that makes up Square C? _____

   c. What is the side length of the large square (written *three* different ways)? _____

Steps for Simplifying Square Roots

1. Find the greatest perfect square that is a factor of the number inside the square root symbol.
2. Rewrite the number inside the square root symbol as the product of the greatest perfect square and the other factor.
3. Take the square root of the perfect square. Remember: When you take the square root of the perfect square, it is no longer inside the square root symbol.
4. Continue this process until you can no longer find a perfect square other than 1 that is a factor of the number inside the square root symbol.

Examples:

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

$$\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}$$

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$$

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$$

Now you try…

$\sqrt{50}$

$\sqrt{200}$

$\sqrt{72}$

$\sqrt{147}$

$\sqrt{128}$

$\sqrt{\frac{1}{4}}$

$\sqrt{\frac{4}{25}}$

$\sqrt{\frac{49}{36}}$

$-\sqrt{36}$

$-\sqrt{8}$

What happens when we apply this same method with a perfect square?
$$\sqrt{100} = \sqrt{25 \cdot 4} = \sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10$$

7.1d Homework: Simplifying Square Roots

Directions: Simplify the following square roots.

1. $\sqrt{4} =$ _____

2. $\sqrt{36} =$ _____

3. $\sqrt{125} =$ _____

4. $\sqrt{216} =$ _____

5. $\sqrt{80} =$ _____

6. $\sqrt{256} =$ _____

7. $\sqrt{28} =$ _____

8. $\sqrt{99} =$ _____

9. $\sqrt{24} =$ _____

10. $\sqrt{12} =$ _____

11. $\sqrt{\frac{1}{64}} =$ _____

12. $\sqrt{\frac{25}{49}} =$ _____

13. $-\sqrt{72} =$ _____

14. $-\sqrt{100} =$ _____

15. $-\sqrt{\frac{121}{144}} =$ _____

7.1e Classwork: Creating Cubes

In the previous lessons, we learned how to find the area of a square given the side length and how to find the side length of a square given the area. In this section, we will study how to find the volume of a cube given its side length and how to find the side length of a cube given its volume.

1. Find the volume of the cube to the left. Describe the method(s) you are using.

2. The cube above is called a perfect cube. A cube is considered a perfect cube if you can arrange smaller unit cubes to build a larger cube. In the example above 27 unit cubes were arranged to build the larger cube shown. Can you build additional perfect cubes to fill in the table below? The first one has been done for you for the cube shown above.

Dimensions	Volume of Cube Exponential Notation (units^3)	Volume of Cube (units^3)	Side Length (units)
$3 \times 3 \times 3$	3^3	27 units^3	3 units

3. Find the side length of a cube with a volume of 27 units^3.

4. Find the side length of a cube with a volume of 125 units^3.

5. Find the side length of a cube with a volume of 30 units^3.

6. Find the side length of a cube with a volume of 100 units^3.

In the previous sections, we learned the following:
- If we are given the side length of a square, s, then its area is s^2.
- If we are given the area of a square, A, then its side length is $\sqrt{A}$.

In this section, we see that:
- If we are given the side length of a cube, s, then its volume is s^3.
- If we are given the volume of a cube, V, then its side length is $\sqrt[3]{V}$. Explain in your own words what $\sqrt[3]{V}$ means.

Directions: Fill in the following blanks.

1. $\sqrt[3]{27}$ = _____ because (____)^3 = 27

2. $\sqrt[3]{64}$ = _____ because (____)^3 = 64

3. $\sqrt[3]{1}$ = _____ because (____)^3 = 1

4. $\sqrt[3]{125}$ = _____

5. $\sqrt[3]{343}$ = _____

6. $\sqrt[3]{\dfrac{1}{216}}$ = _____

7. $\sqrt[3]{\dfrac{1}{1000}}$ = _____

8. $\sqrt[3]{\dfrac{8}{125}}$ = _____

7.1f Class Activity: Revisiting the Number Line

Directions: Show the length of the following numbers on the number line below. Use the grid on the following page to construct lengths where needed and transfer those lengths onto the number line. Then answer the questions that follow. **Note:** On the grid, a horizontal or vertical segment joining two dots has a length of 1. On the number line, the unit length is the same as the unit length on the dot grid.

$A: \sqrt{25}$ $B: \sqrt{2}$ $C: \sqrt{8}$ $D: 2\sqrt{2}$ $E: \sqrt{5}$ $F: 2\sqrt{5}$
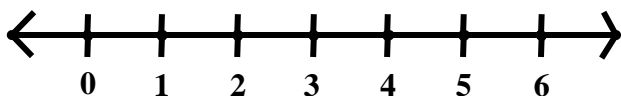
1. Use the number line to write a decimal approximation for $\sqrt{2}$. Verify your estimate with a calculator.

2. Would 1.41 be located to the right or to the left of $\sqrt{2}$ on the number line?

3. Describe and show how you can put $-\sqrt{2}$ on the number line. Write the decimal approximation for $-\sqrt{2}$.

4. Describe and show how you can put $(2 + \sqrt{2})$ on the number line. Estimate the value of this expression.

5. Describe and show how you can put $(2 - \sqrt{2})$ on the number line. Estimate the value of this expression.

6. Describe and show how you can put $2\sqrt{2}$ on the number line. Estimate the value of this expression.

7. Use the number line to write a decimal approximation for $\sqrt{5}$.

8. Would 2.24 be located to the right or to the left of $\sqrt{5}$ on the number line?

9. Describe and show how you can put $1 + \sqrt{5}$ on the number line. Estimate the value of this expression.
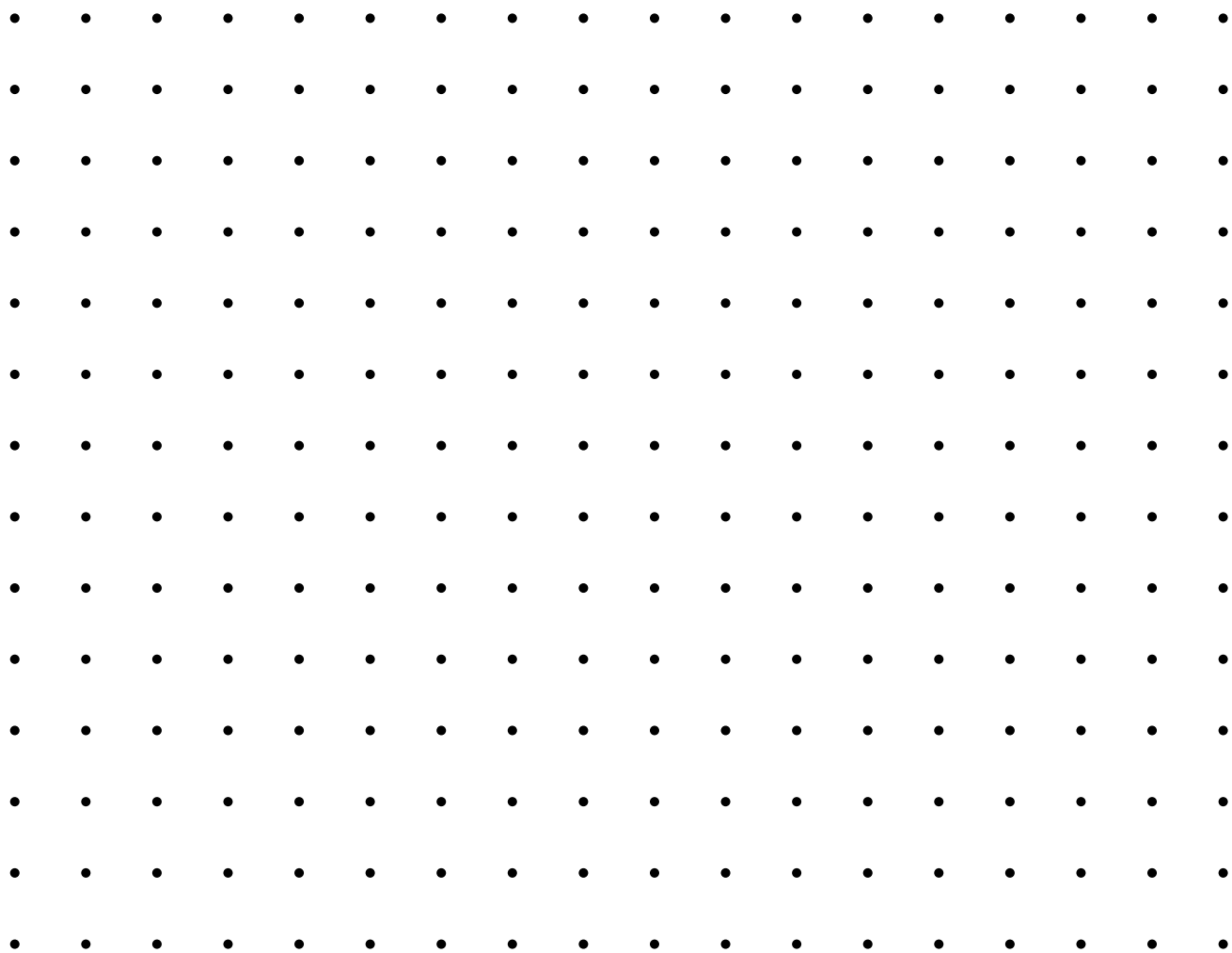
7.1g Self-Assessment: Section 7.1

Consider the following skills/concepts. Rate your comfort level with each skill/concept by checking the box that best describes your progress in mastering each skill/concept.

Skill/Concept	Beginning Understanding	Developing Skill and Understanding	Deep Understanding, Skill Mastery
1. Understand the relationship between the side length of a square and its area.			
2. Understand the relationship between the side length of a cube and its volume.			
3. Understand what is meant by the square root and cube root symbols.			
4. Construct physical lengths of perfect and non-perfect squares.			
5. Evaluate the square roots of small perfect squares and the cube roots of small perfect cubes.			
6. Simplify square roots.			

Section 7.2: Rational and Irrational Numbers

Section Overview:
This section begins with a review of the different types of rational numbers (whole numbers, fractions, integers, etc.) and why a need arose for them. Students associate these different types of rational numbers to points on the number line. Students then review how to change fractions into decimals and during the process, are reminded that the decimal expansion of all rational numbers either terminates or repeats eventually. From here, students review how to express terminating decimals as fractions and learn how to express repeating decimals as fractions by setting up and solving a system of equations. This skill allows them to show that all decimals that either terminate or repeat can be written as a fraction and therefore fit the definition of a rational number. After this work with rational numbers, students investigate numbers whose decimal expansion does not terminate or repeat: irrational numbers. With this knowledge, students classify numbers as rational and irrational. Students learn different methods for approximating the value of irrational numbers, zooming in to get better and better approximations of the number. They then use these approximations to estimate the value of expressions containing irrational numbers. Lastly, students compare and order rational and irrational numbers.

Concepts and Skills to Master:
By the end of this section, students should be able to:

1. Know that real numbers that are not rational are irrational.
2. Show that rational numbers have decimal expansions that either terminate or repeat eventually.
3. Convert a repeating decimal into a fraction.
4. Understand that all non-perfect square roots and cube roots are irrational.
5. Understand that the decimal expansions of irrational numbers are approximations.
6. Understand that the real number line represents a continuum of numbers and that all real numbers can be thought of as points on the number line.
7. Show the location (or approximate location) of real numbers on the real number line.
8. Approximate the value of irrational numbers, zooming in to get better and better approximations.
9. Estimate the value of expressions containing irrational numbers.
10. Compare and order rational and irrational numbers.

7.2a Classwork: The Rational Number System
Our number system has evolved over time. On the following pages, you will review the numbers that make up the **rational number system**.

Whole Numbers:
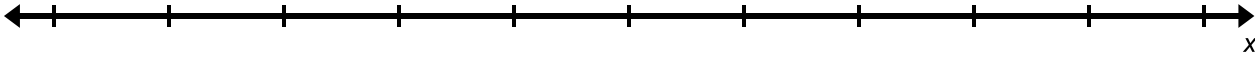
Integers:
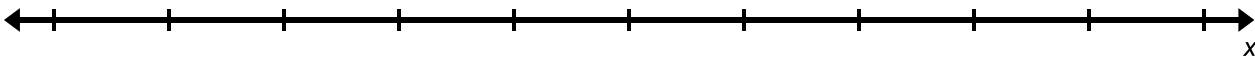
Fractions:
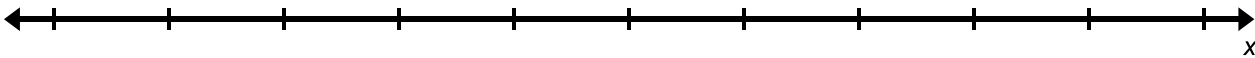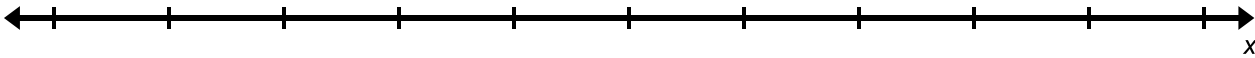
Over the years, you have expanded your knowledge of the number system, gradually incorporating the different types of numbers mentioned above. These numbers are all part of the **rational number system**.

A **rational number** is any number that can be expressed as a quotient $\frac{p}{q}$ of two integers where q does not equal 0.

1. Begin to fill out the table below with different types of **rational numbers** you know about so far and give a few examples of each. You will continue to add to this list throughout this section.

Types of Rational Numbers	Examples

2. Change the following rational numbers into decimals **without** the use of a calculator.

a. $\frac{1}{2}$	b. $\frac{3}{5}$
c. $\frac{3}{8}$	d. $\frac{1}{3}$

e. $\frac{4}{15}$	f. $\frac{1}{7}$

3. What do you notice about the decimal expansion of any rational number? Why is this true?

4. Revisit question #1 on page 49. Add additional types of rational numbers to your list.

7.2a Homework: The Rational Number System

1. Give 5 examples of numbers that are rational.

2. Change the following rational numbers into decimals **without** the use of a calculator.

a. $\frac{1}{5}$	b. $\frac{3}{4}$
c. $\frac{5}{8}$	d. $\frac{2}{3}$
e. $\frac{2}{9}$	f. $\frac{3}{11}$

7.2b Classwork: Expressing Decimals as Fractions

As we discovered in the previous section, when we converted fractions into decimals, the result was either a **terminating or repeating decimal**. We added terminating and repeating decimals to our list of different types of rational numbers.

If we are given a terminating or repeating decimal, we need a method for changing them into a fraction in order to prove that they fit the definition of a rational number.

In 7th grade, you learned how to convert terminating decimals into fractions. Here are a few examples:

$$0.3 = \frac{3}{10}$$

$$0.25 = \frac{25}{100} = \frac{1}{4}$$

$$0.375 = \frac{375}{1000} = \frac{3}{8}$$

$$-2.06 = -2\frac{6}{100} = -2\frac{3}{50} = -\frac{103}{50}$$

Now you try a few...

$0.4 =$

$0.05 =$

$0.275 =$

$1.003 =$

So, how do we express a repeating decimal as a fraction? For example, how would you convert the repeating decimal $0.\overline{45}$ into a fraction? Try in the space below.

We can use a **system of two linear equations** to convert a repeating decimal into a fraction. Let's look at an example:

Example 1:

The decimal $0.\overline{3}$ is a repeating decimal that can be thought of as $0.33333\ldots$ where the "…" indicates that the 3s repeat forever. If they repeat forever, how can we write this number as a fraction? Here's a trick that will eliminate our repeating 3s.

Let a represent our number $a = 0.\overline{3}$.

Multiply both sides of the equation by 10 which would give us a second equation $10a = 3.\overline{3}$.

Now we have the following two equations:

$10a = 3.\overline{3}$

$a = 0.\overline{3}$

Let's expand these out:

$10a = 3.333333333333\ldots\ldots$

$a = 0.333333333333\ldots\ldots$

What will happen if we subtract the second equation from the first? Let's try it (remembering to line up the decimals):

$\ 10a = 3.333333333333\ldots\ldots$

$-\ \ a = 0.333333333333\ldots\ldots$

_____

$\ \ \ 9a = 3$

$\ \ \ a = \dfrac{3}{9}$ (Divide both sides by 9)

$\ \ \ a = \dfrac{1}{3}$ (Simplify the fraction)

Example 2:
The decimal $0.\overline{54}$ is a repeating decimal that can be thought of as $0.54545454\ldots$ where the "…" indicates that the 54 repeats forever. Let's see how to express this as a fraction.

Let a represent our number $a = 0.\overline{54}$.
Multiply both sides of the equation by 100 this time which would give us a second equation $100a = 54.\overline{54}$.

Now we have the following two equations:
$100a = 54.\overline{54}$
$a = 0.\overline{54}$

Again, let's expand these out:

$100a = 54.5454545454\ldots\ldots$
$a = 0.5454545454\ldots\ldots$

Next, subtract the second equation from the first (again, remembering to line up the decimals):

$100a = 54.5454545454\ldots\ldots$
$-a = 0.5454545454\ldots\ldots$
_____

$99a = 54$
$a = \dfrac{54}{99}$ (Divide both sides by 99)
$a = \dfrac{6}{11}$ (Simplify the fraction)

Why do you think we multiplied the second example by 100 instead of 10 as we did in the first example? What would have happened if we had multiplied by 10 in example 2? Try it below and see.

Now you try:

Example 3: Change the decimal $0.\overline{123}$ into a fraction.

Example 4: Change the decimal $4.\overline{1}$ into a fraction.

Example 5: Change the decimal $2.0\overline{15}$ into a fraction.

7.2b Homework: Expressing Decimals as Fractions

Directions: Change the following decimals into fractions.

1. 0.6	2. 0.02
3. 1.25	4. −0.064
5. $0.\overline{5}$	6. $0.\overline{45}$
7. $0.08\overline{3}$	8. $5.2\overline{6}$

7.2c Classwork: Expanding Our Number System

So, are all numbers **rational numbers?** Are there any numbers that **cannot** be expressed as a quotient $\frac{p}{q}$ of two integers?

What about $\sqrt{2}$? Can you write $\sqrt{2}$ as a fraction? Why or why not?

Numbers like $\sqrt{2}$, which do not have a terminating or repeating decimal expansion, are **irrational numbers**. Irrational numbers **cannot** be expressed as a quotient.

In the space below, can you think of other numbers that are likely to be irrational?

Rational and Irrational numbers together form the set of **real numbers**. Real numbers can be thought of as points on an infinitely long line called the number line.

Real Number System

Directions: Classify the following numbers as rational or irrational and provide a justification.

Number	Rational or Irrational	Justification
1. $\frac{2}{3}$		
2. 0.25		
3. -2		
4. $\sqrt{5}$		
5. 10		
6. 0		
7. $\sqrt{10}$		
8. $\sqrt{36}$		
9. $-\sqrt{121}$		
10. $2\frac{1}{2}$		
11. $0.08\overline{3}$		
12. 0.0825		
13. $\frac{10}{13}$		
14. π		
15. -3π		
16. $0.26\overline{54}$		
17. $\sqrt[3]{27}$		
18. 1.2122122212222…		
19. $\sqrt[3]{30}$		
20. $\frac{\sqrt{2}}{2}$		
21. The side length of a square with an area of 2		
22. The side length of a square with an area of 9		
23. The number half-way between 3 and 4		
24. The number that represents a loss of 5 yards		

7.2c Homework: Expanding Our Number System

Directions: Classify the following numbers as rational or irrational and provide a justification.

Number	Rational or Irrational	Justification
1. $\sqrt{2}$		
2. $\sqrt{1}$		
3. $\frac{1}{3}$		
4. -157		
5. $4\frac{1}{9}$		
6. -0.375		
7. $-\sqrt{5}$		
8. $0.\overline{2}$		
9. $\sqrt[3]{125}$		
10. $-\sqrt{81}$		
11. $-2.2\overline{4}$		
12. 2π		
13. The side length of a square with an area of 49		
14. The side length of a square with an area of 1		
15. The side length of the side of a square with an area of 5		
16. The side length of a square with an area of 24		
17. The number half-way between 0 and -1		
18. The number that represents 7 degrees below 0.		

19. Give your own example of a rational number.

20. Give your own example of an irrational number.

Directions: The table below contains statements about rational and irrational numbers. If the statement is true, put a check in the box. If the statement is not true, write a correct statement.

Statement	Check if True or Correct Statement
21. You can show the exact decimal expansion of the side length of a square with an area of 5 square units.	
22. You can construct and show the length $\sqrt{5}$ on a number line.	
23. Square roots of numbers that are perfect squares are rational.	
24. The number 0.256425642564 ... is rational.	
25. You can always use a calculator to determine whether a number is rational or irrational by looking at its decimal expansion.	
26. The number $0.\overline{6}$ is irrational because its decimal expansion goes on forever.	
27. The number half-way between 3 and 4 is rational.	
28. You can build a perfect cube with 36 unit cubes.	
29. If you divide an irrational number by 2, you will still have an irrational number.	
30. The side length of a cube made of 64 unit blocks is irrational.	

Make up **two** of your own statements that are **true** about rational or irrational numbers.

7.2d Classwork: Approximating the Value of Irrational Numbers

So far, we have seen that we can show the location of an irrational number on the number line. We also know that we cannot show the entire decimal expansion of an irrational number because it is infinitely long and there is no pattern (as far as we know). However, we can come up with good approximations for the numerical value of an irrational number.
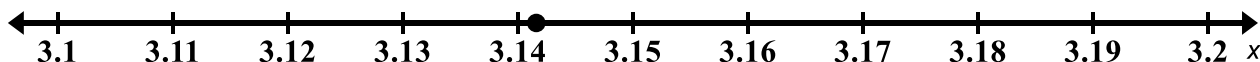
The decimal expansion for π to eight decimal places is 3.14159265… On the number line, we know that π lies somewhere between 3 and 4:
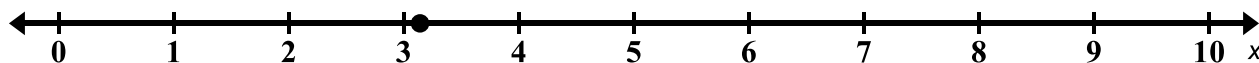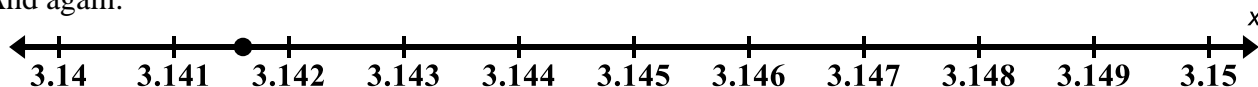

We can zoom in on the interval between 3 and 4 and narrow in on where π lies:
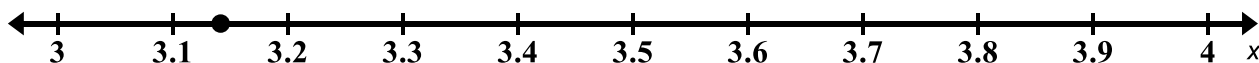
And if we zoom in again on the interval from 3.1 to 3.2:

And again:

We can imagine continuing this process of zooming in on the location of π on the number line, each time narrowing its possible location by a factor of 10.

Once we have an approximation for an irrational number, we can approximate the value of expressions that contain that number.

For example, suppose we were interested in the approximate value of 2π? We can use our approximations of π from above to approximate the value of 2π to different degrees of accuracy:

Because π is between 3 and 4, 2π is between _____ and _____.
Because π is between 3.1 and 3.2, 2π is between _____ and _____.
Because π is between 3.14 and 3.15, 2π is between _____ and _____.
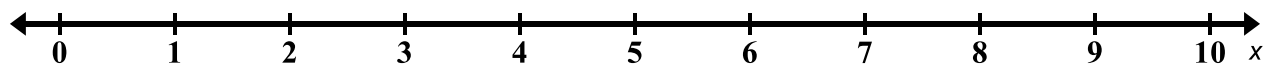Because π is between 3.141 and 3.142, 2π is between _____ and _____.

Check the value of 2π on your calculator. How are we doing with our approximations of 2π?

We can use a method of **guess and check** to give us an estimate of the numerical value of an irrational number that is correct up to as many decimal points as we need.
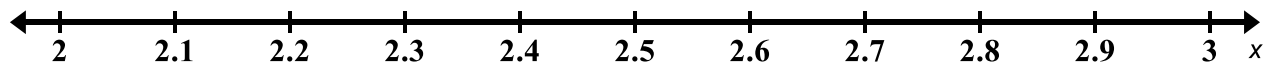
Directions: Approximate the value of the following irrational numbers to the indicated degrees of accuracy. You can use your calculator for the following questions but do not use the square root key.

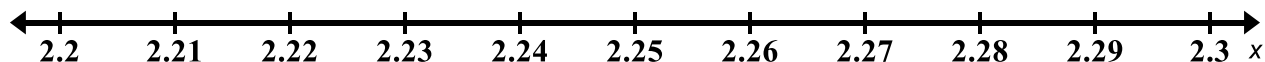1. Between which two integers does $\sqrt{5}$ lie?

    a. Which integer is it closest to?
    b. Show its approximate location on the number line below.

```
◄──┼────┼────┼────┼────┼────┼────┼────┼────┼────┼───►
   0    1    2    3    4    5    6    7    8    9    10  x
```

    c. Now find $\sqrt{5}$ accurate to one decimal place. Show its approximate location on the number line below.

```
◄──┼────┼────┼────┼────┼────┼────┼────┼────┼────┼───►
   2   2.1  2.2  2.3  2.4  2.5  2.6  2.7  2.8  2.9   3   x
```

    d. Now find $\sqrt{5}$ accurate to two decimal places. Show its approximate location on the number line below.

```
◄──┼────┼────┼────┼────┼────┼────┼────┼────┼────┼───►
  2.2  2.21 2.22 2.23 2.24 2.25 2.26 2.27 2.28 2.29 2.3  x
```

    e. Use your work from above to approximate the value of the expression $2 + \sqrt{5}$ to increasing levels of accuracy.

2. Between which two integers does $\sqrt{15}$ lie?

    a. Which integer is it closest to?
    b. Show its approximate location on the number line below.

    c. Now find $\sqrt{15}$ accurate to one decimal place. Show its approximate location on the number line below.
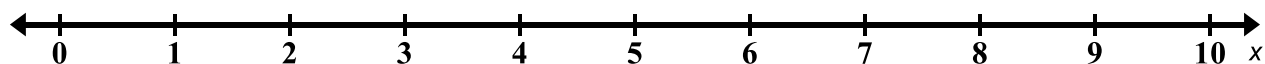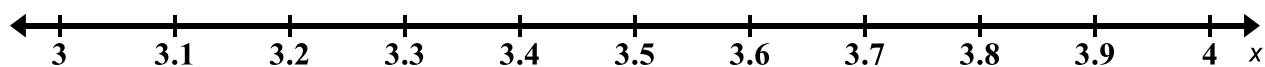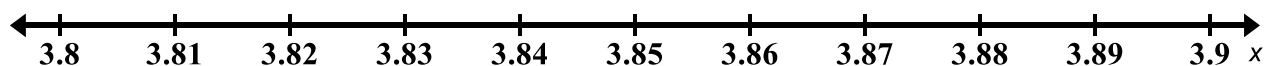

    d. Now find $\sqrt{15}$ accurate to two decimal places. Show its approximate location on the number line below.
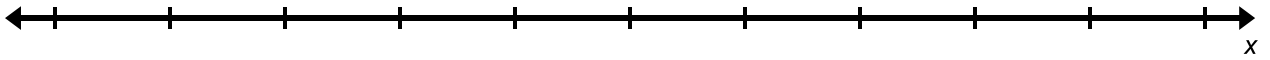

    e. Use your work from above to approximate the value of the expression $4\sqrt{15}$ to increasing levels of accuracy.

3. Repeat the process above to find $\sqrt{52}$ accurate to two decimal places. Place your numbers on the number lines provided each time you increase the degree of accuracy of your estimate.
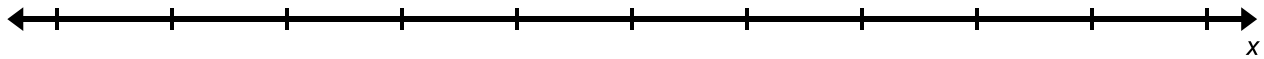
a. To the nearest **whole number**:

b. To the nearest **tenth**:

c. To the nearest **hundredth**:

d. Use your work from above to approximate the value of $3 + \sqrt{52}$ to increasing levels of accuracy.

e. Use your work from above to approximate the value of $2\sqrt{52}$ to increasing levels of accuracy.

Directions: Solve the following problems. Again, do not use the square root key on your calculator.

4. A hospital has asked a medical supply company to manufacture intravenous tubing (IV tubing) that has a minimum opening of 7 square millimeters and a maximum opening of 7.1 square millimeters for the rapid infusion of fluids. The medical design team concludes that the radius of the tube opening should be 1.5 mm. Two supervisors review the design team's plans, each using a different estimation for π.

   Supervisor 1: Uses 3 as an estimation for π

   Supervisor 2: Uses 3.1 as an estimation for π

   The supervisors tell the design team that their designs will not work. The design team stands by their plans and tells the supervisors they are wrong. Who is correct and why? Recall that the formula for the area of a circle is $A = \pi r^2$.

5. A square field with an area of 2,000 square ft. is to be enclosed by a fence. Three contractors are working on the project and have decided to purchase slabs of pre-built fencing. The slabs come in pieces that are 5-ft. long.

   - Keith knows that $\sqrt{2000}$ is between 40 and 50. Trying to save as much money as possible, he estimates on the low side and concludes that they will need 160 feet of fencing. Therefore, he concludes they should purchase 32 slabs of the material.

   - Jose also knows that $\sqrt{2000}$ is between 40 and 50 but he is afraid that using Keith's calculations, they will not have enough fencing. He suggests that they should estimate on the high side and buy 200 feet of fencing to be safe. Therefore, he concludes they should purchase 40 slabs of material.

   Keith and Jose begin to argue. Sam jumps in and says, "I have a way to make you both happy – we will purchase enough material to enclose the entire field **and** we will minimize the amount of waste." What do you think Sam's suggestion is and how many slabs will be purchased using Sam's rationale?
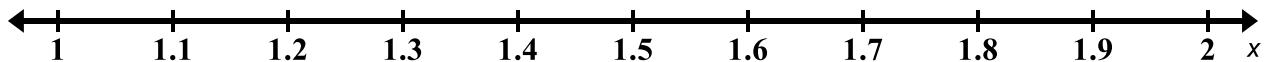
7.2d Homework: Approximating the Value of Irrational Numbers

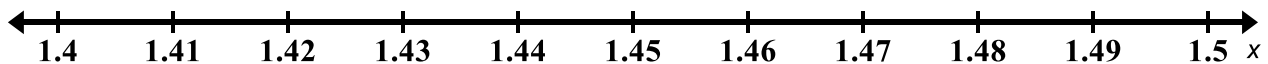1. Between which two integers does $\sqrt{2}$ lie?

    a. Which integer is it closest to?

    b. Show its approximate location on the number line below.
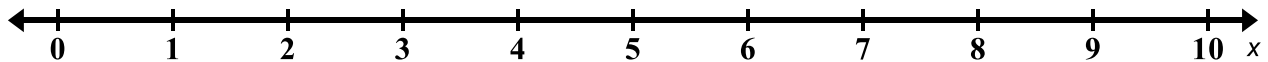
    c. Now find $\sqrt{2}$ accurate to one decimal place. Show its approximate location on the number line below.

    d. Now find $\sqrt{2}$ accurate to two decimal places. Show its approximate location on the number line below.
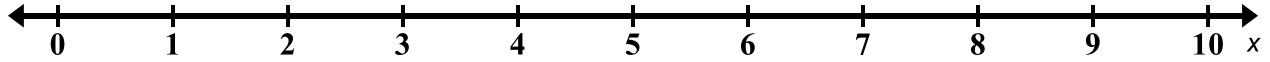

    e. Estimate the value of the expression $2 + \sqrt{2}$ to increasing levels of accuracy.

    f. Estimate the value of the expression $2\sqrt{2}$ to increasing levels of accuracy.

2. Between which two integers does $\sqrt{40}$ lie?

    a. Which integer is it closest to?
    b. Show its approximate location on the number line below.

    c. Now find $\sqrt{40}$ accurate to one decimal place. Show its approximate location on the number line below.

    d. Now find $\sqrt{40}$ accurate to two decimal places. Show its approximate location on the number line below.

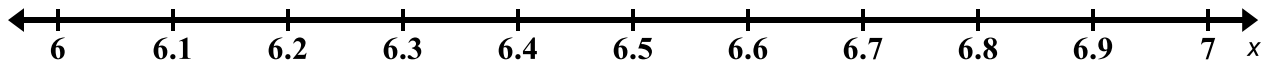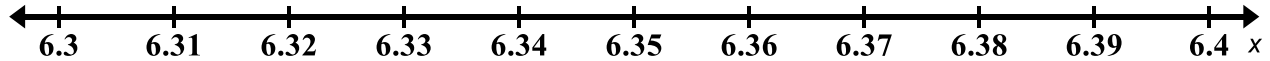   e. Estimate the value of the expression $2\sqrt{40}$ to increasing levels of accuracy.

3. Repeat the process above to find $\sqrt{60}$ accurate to two decimal places. Place your numbers on the number lines provided each time you increase the degree of accuracy of your estimate.

   a. To the nearest **whole number**:
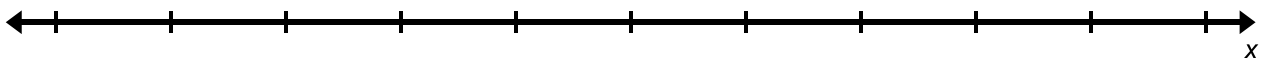
   b. To the nearest **tenth**:
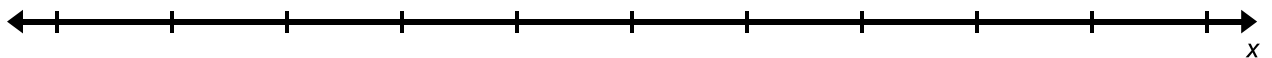
   c. To the nearest **hundredth**:
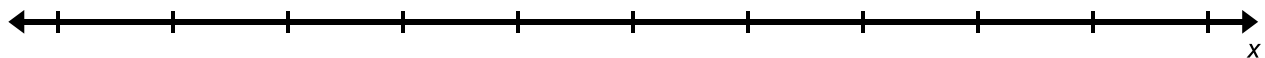
   d. Use your work from above to approximate the value of $\sqrt{60} - 5$ to increasing levels of accuracy.

   e. Use your work from above to approximate the value of $1 + \sqrt{60}$ to increasing levels of accuracy.

4. Use the estimations on page 61 to estimate the value of the following expressions to increasing levels of accuracy. You can use your calculator but don't use the square key or the π key.

    a. π^2

    b. 10π

    c. $3 + \pi$

7.2e Classwork: Comparing and Ordering Real Numbers

Directions: Do **not** use a calculator for the following problems. Any calculations you may need are given in the problem.

1. Put the following numbers in order from **least to greatest**. Note that $8.5^2 = 72.25$.
   $\sqrt{80}, 8, 9, 8.5, \sqrt{62}$

2. Put the following numbers in order from **least to greatest**. Note that $3.5^2 = 12.25$.
   $-\sqrt{13}, -3, -4, -3.5$

3. Use the following calculations to answer the questions below.

   $2.2^2 = 4.84$
   $2.3^2 = 5.29$
   $2.23^2 = 4.9729$
   $2.24^2 = 5.0176$

   a. Put the following numbers in order from **least to greatest**.
   $\sqrt{5}, \frac{5}{2}, 2.2$, the side length of a square with an area of 4

   b. Find a number between 2.2 and $\sqrt{5}$.

   c. Find an irrational number that is smaller than all of the numbers in part a.

4. Use the following calculations to answer the questions below.

   $6.48^2 = 41.9904$
   $6.5^2 = 42.25$

   a. Put the following numbers in order from **least to greatest**.
   $\sqrt{50}, 6, 7, 6.5, \sqrt{42}$

   b. Find a rational number that is smaller than all of the numbers in part a.

   c. Find an irrational number that is smaller than all of the numbers in part a.

   d. Find a number between $\sqrt{42}$ and 6.5.

5. Use the following calculations to answer the questions below.

$2.44^2 = 5.9536$
$2.45^2 = 6.0025$
$2.449^2 = 5.997601$

    a. Put the following numbers in order from **least to greatest**.
$\sqrt{6}, 2.44, 2.\overline{4}, 2.5$, the side length of a square with an area of 9

    b. Find an irrational number that is between 0 and the smallest number from part a.

    c. Find a number that is between 2.44 and $\sqrt{6}$.

6. Use the approximations of π and the calculations given below to answer the questions below.
π is between 3 and 4
π is between 3.1 and 3.2
π is between 3.14 and 3.15
π is between 3.141 and 3.142

$3.15^2 = 9.9225$

    a. Find a number that is between 3 and π.
    b.

    c. Find a number that is between 3.14 and π.

    d. Which is larger and why? $(\pi + 5)$ or 8

    e. Which is larger and why? $(10 - \pi)$ or 7

    f. Which is larger and why? 2π or 6.2

    g. Which is larger and why? π^2 or 10

7.2e Homework: Comparing and Ordering Real Numbers

Directions: Do **not** use a calculator for the following problems. Any calculations you may need are given in the problem.

1. Give an example of a rational number between $\sqrt{9}$ and $\sqrt{16}$.

2. Give an example of an irrational number between 8 and 9.

3. Use the following calculations to answer the questions below.

   $1.41^2 = 1.9881$
   $1.42^2 = 2.0164$

   a. Put the following numbers in order from **least to greatest**.
      $\sqrt{2}, 1.41, 1.4, 1\frac{1}{2}, 1.4\overline{2}$

   b. Find a number between 1.4 and $1\frac{1}{2}$.

4. Use the following approximations and calculations to answer the questions below.

   π is between 3.14 and 3.15

   $3.1^2 = 9.61$
   $3.2^2 = 10.24$
   $3.16^2 = 9.9856$
   $3.17^2 = 10.0489$

   a. Put the following numbers in order from **least to greatest**.
      $\sqrt{10}, 3\frac{1}{10}, 3.\overline{1}, \pi$, side length of a square with an area of 9

   b. Find a number between $3\frac{1}{10}$ and $3.\overline{1}$.

   c. Find a number between 3.1 and $\sqrt{10}$.

5. The number e is a famous irrational number that is one of the most important numbers in mathematics. Use the approximations of e and the calculations given below to answer the questions below.

e is between 2 and 3

e is between 2.7 and 2.8

e is between 2.71 and 2.72

e is between 2.718 and 2.719

    a. Find a number that is between 2 and e.

    b. Find a number that is between e and 2.8.

    c. Which is larger and why? $(e + 10)$ or 13

    d. Which is larger and why? $(6 - e)$ or 4

    e. Which is larger and why? $2e$ or 5.4

    f. Which is larger and why? e^2 or 9

6. Put the following numbers in order from **least to greatest**. Note that $6.2^2 = 38.44$ and $6.4^2 = 40.96$

$$-\sqrt{40}, -7, -6, -6.2, -6.4, -6\frac{1}{2}$$

7.2f Self-Assessment: Section 7.2

Consider the following skills/concepts. Rate your comfort level with each skill/concept by checking the box that best describes your progress in mastering each skill/concept.

Skill/Concept	Beginning Understanding	Developing Skill and Understanding	Deep Understanding, Skill Mastery
1. Know that real numbers that are not rational are irrational.			
2. Show that rational numbers have decimal expansions that either terminate or repeat.			
3. Convert a repeating decimal into a fraction.			
4. Understand that all non-perfect square roots and cube roots are irrational.			
5. Understand that the decimal expansions of irrational numbers are approximations.			
6. Understand that the real number line represents a continuum of numbers and that all real numbers can be thought of as points on the number line.			
7. Show the location (or approximate location) of real numbers on the real number line.			
8. Approximate the value of irrational numbers, zooming in to get better and better approximations.			
9. Estimate the value of expressions containing irrational numbers.			
10. Compare and order rational and irrational numbers.			